Tangentsand Normals

1 Marks Questions

1. Find the slope of tangent to the curve $y = 3x^2 - 6$ at the point on it whose x-coordinate is 2.
All India 2009C

Firstly, differentiate the given function with respect to x and then determine the value of $\frac{dy}{dx}$ at $x = 2$.

Given, $y = 3x^2 - 6$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6x$$

At $x = 2$, slope of tangent

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = 6\,(2) = 12$$

$\therefore$ Required slope = 12 **(1)**

2. Find the slope of tangent to the curve $y = 3x^2 - 4x$ at point whose x-coordinate is 2. Delhi 2009C

Given, $y = 3x^2 - 4x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6x - 4$$

At $x = 2$, slope of tangent

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = 6\,(2) - 4 = 12 - 4 = 8$$

$\therefore$ Required slope = 8 **(1)**

3. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 1$. All India 2008C

Do same as Que. 2. **[Ans.** 8]

4. For the curve $y = 3x^2 + 4x$, find the slope of tangent to the curve at point, where x-coordinate is -2. Delhi 2008C

Do same as Que. 2. **[Ans.**– 8]

4 Marks Questions

5. Find the equations of the tangent and normal to the curves $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

Delhi 2014

Given, $x = a\sin^3\theta$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(3\sin^2\theta\cos\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

and $y = a\cos^3\theta$ **(1)**

On differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = -3a\cos^2\theta\sin\theta$$

Then, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\cot\theta$$ **(1)**

At $\theta = \frac{\pi}{4}$, $\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\cot\frac{\pi}{4}$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -1$$

Also, at $\theta = \frac{\pi}{4}$, $x = a\left(\sin\frac{\pi}{4}\right)^3$, $y = a\left(\cos\frac{\pi}{4}\right)^3$

$$\Rightarrow \quad x = a\left(\frac{1}{2}\right)^{3/2}, y = a\left(\frac{1}{2}\right)^{3/2}$$

$$\left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

Now, equation of tangent at the point $\left[\frac{a}{(2)^{3/2}}, \frac{a}{(2)^{3/2}}\right]$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad Y - \frac{a}{(2)^{3/2}} = (-1)\left[X - \frac{a}{2^{3/2}}\right]$$

$$\Rightarrow \quad Y + X = \frac{2a}{(2)^{3/2}}$$

$$\Rightarrow \quad Y + X = \frac{a}{\sqrt{2}} \qquad \textbf{(1)}$$

$$\Rightarrow \quad X + Y - \frac{a}{\sqrt{2}} = 0$$

Also, slope of normal $= \frac{-1}{\text{Slope of tangent}}$

$\Rightarrow$ Slope of normal $= 1$

$\therefore$ Equation of normal at the point $\left[\frac{a}{(2)^{3/2}}, \frac{a}{(2)^{3/2}}\right]$ is

$$Y - \frac{a}{(2)^{3/2}} = (1)\left[X - \frac{a}{2^{3/2}}\right]$$

$$\Rightarrow \quad X - Y = 0 \qquad \textbf{(1)}$$

6. Find the equations of the tangent and normal to the curves $\frac{x^2}{d^2}-\frac{y^2}{b^2}=1$ at the point $(\sqrt{2}a, b)$. 2014

The equation of the given curve is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

On differentiating both sides w.r.t. x, we get

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y} \quad (1)$$

$\therefore$ Slope of the tangent at point $(\sqrt{2}a, b)$ is

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(\sqrt{2}a,\, b)} = \frac{\sqrt{2}ab^2}{ba^2} = \frac{\sqrt{2}b}{a} \qquad \textbf{(1)}$$

Hence, the equation of the tangent at point $(\sqrt{2}a, b)$ is

$$y - b = \frac{\sqrt{2}b}{a}(x - \sqrt{2}a)$$

$$\Rightarrow \quad a(y - b) = \sqrt{2}\, b\,(x - \sqrt{2}a)$$

$$\Rightarrow \quad ay - ab = \sqrt{2}\, bx - 2ab$$

$$\Rightarrow \quad \sqrt{2}bx - ay - ab = 0$$

Now, the slope of the normal at point $(\sqrt{2}a, b)$

$$= \frac{-1}{\text{Slope of tangent}} = \left[\frac{-a^2y}{b^2x}\right]_{(\sqrt{2}a,\, b)} = -\frac{a}{\sqrt{2}b}$$

(1)

Hence, the equation of the normal at point $(\sqrt{2}a, b)$ is

$$(y - b) = -\frac{a}{\sqrt{2}b}(x - \sqrt{2}a)$$

$$\Rightarrow \quad \sqrt{2}\, b\,(y - b) = -a\,(x - \sqrt{2}a)$$

$$\Rightarrow \quad \sqrt{2}\, b\, y - \sqrt{2}\, b^2 = -ax + \sqrt{2}\, a^2$$

$$\Rightarrow \quad ax + \sqrt{2}by - \sqrt{2}\,(a^2 + b^2) = 0 \qquad \textbf{(1)}$$

7. Find the points on curve $y = x^3 - 11x + 5$ at which equation of tangent is $y = x - 11$.

Delhi 2012C; HOTS

Firstly, find the slope of given curve and given tangent, then equate them to get value x. Put value of x in given curve to find required points.

Given equation of curve is

$$y = x^3 - 11x + 5 \quad \text{...(i)}$$

Slope of the tangent to the given curve is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 11 \quad \text{...(ii)} \ \mathbf{(1)}$$

Also, slope of the tangent $y = x - 11$ is 1.

$$\therefore \quad \frac{dy}{dx} = 1$$

$$\Rightarrow \quad 3x^2 - 11 = 1 \quad \text{[from Eq. (i)]} \ \mathbf{(1)}$$

$$\Rightarrow \quad 3x^2 = 12 \Rightarrow x^2 = 4$$

$$\therefore \quad x = \pm 2 \quad \mathbf{(1)}$$

Then, from Eq. (i)

When $x = 2$

then $y = (2)^3 - 11(2) + 5$

$= 8 - 22 + 5 = -9$

When $x = -2$, then

$y = (-2)^3 - 11(-2) + 5$

$= -8 + 22 + 5 = 19$

Hence, the required points on the curve are (2, –9) and (–2, 19). **(1)**

8. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which tangent is parallel to X-axis. **Delhi 2011**

We know that, when a tangent is parallel to X-axis, then $\frac{dy}{dx} = 0$. So, put $\frac{dy}{dx} = 0$ and find value of x from it. Then, put this value of x in the equation of the given curve and find value of y.

Given equation of curve is

$$x^2 + y^2 - 2x - 3 = 0 \quad \ldots(i)$$

Now, differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2-2x}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-x}{y} \quad \textbf{(1)}$$

We know that, when a tangent to the curve is parallel to X-axis, then $\frac{dy}{dx} = 0$. **(1)**

On putting $\frac{dy}{dx} = 0$, we get

$$1 - x = 0 \Rightarrow x = 1 \quad \textbf{(1)}$$

Now, on putting $x = 1$ in Eq. (i), we get

$$1 + y^2 - 2 - 3 = 0$$

$$\Rightarrow \quad y^2 - 4 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Hence, required points are (1, 2) and (1, –2).

9. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to y-coordinate of the point. **HOTS; Foreign 2011**

Given, a tangent is equal to y- coordinate of the point, so put $\frac{dy}{dx} = y$ and find value of x from it. Then, put this value of x in the equation of the given curve and find the value of y.

Given equation of curve is $y = x^3$. ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2$$

$\therefore$ Slope of tangent $= \frac{dy}{dx} = 3x^2$ **(1)**

Now, given that slope of tangent

= y- coordinate of the point

$$\Rightarrow \quad \frac{dy}{dx} = y$$

$$\Rightarrow \quad 3x^2 = y \quad \left[\because \frac{dy}{dx} = 3x^2\right]$$

$$\Rightarrow \quad 3x^2 = x^3 \quad [\because y = x^3]$$

$$\Rightarrow \quad 3x^2 - x^3 = 0 \Rightarrow x^2(3 - x) = 0$$

$$\Rightarrow \quad \text{Either } x^2 = 0 \text{ or } 3 - x = 0$$

$$\therefore \quad x = 0, 3$$ **(1)**

Now, on putting $x = 0, 3$ in Eq. (i), we get

$$y = (0)^3 = 0 \quad [\text{at } x = 0]$$

and $$y = (3)^3 = 27 \quad [\text{at } x = 3]$$ **(1)**

Hence, the required points are (0, 0) and (3, 27). **(1)**

10. Find the equation of tangent to curve $x = \sin 3t$, $y = \cos 2t$ at $t = \frac{\pi}{4}$. All India 2011C, 2008

We know that, the equation of tangent at the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

where, $$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} \quad \text{...(i)}$$ **(1/2)**

Now, given $x = \sin 3t$ ...(ii)

$\therefore$ $$\frac{dx}{dt} = 3\cos 3t \quad \text{[differentiate w.r.t. } t\text{]}$$

and $y = \cos 2t$ ...(iii)

$\therefore$ $$\frac{dy}{dt} = -2\sin 2t \quad \text{[differentiate w.r.t. } t\text{]}$$

Then, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{3\cos 3t}$$ **(1)**

On putting $t = \frac{\pi}{4}$, we get

$$m = \left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{-2\sin\frac{\pi}{2}}{3\cos\frac{3\pi}{4}} = \frac{-2}{-\frac{3}{\sqrt{2}}}$$

$$\left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\right]$$

$\Rightarrow$ $$m = \frac{2\sqrt{2}}{3}$$ **(1)**

Also, to find (x_1, y_1), we put $t = \frac{\pi}{4}$ in Eqs. (ii) and (iii), we get

$$x_1 = \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and $\quad y_1 = \cos\frac{\pi}{2} = 0$

$\therefore \ (x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$ **(1/2)**

Now, on putting $(x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$

and $\quad m = \frac{2\sqrt{2}}{3}$ in Eq. (i), we get

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow 3y = 2\sqrt{2}x - \frac{2}{3}$$

Hence, required equation of tangent is

$6\sqrt{2}x - 9y - 2 = 0.$ **(1)**

11. Find the equations of tangents to the curve $y = (x^2 - 1)(x - 2)$ at the points, where the curve cuts the X-axis. **All India 2011C**

The curve cuts the X-axis, so put $y=0$ and get the corresponding values of x. Further, differentiate and determine the slopes at different points. And then use the formula $y-y_1=m(x-x_1)$ to determine the equation of tangent.

Given equation of the curve is

$$y=(x^2-1)(x-2) \quad \text{...(i)}$$

Since, the curve cuts the X-axis, so at that point y- coordinate will be zero.

So, on putting $y=0$, we get

$$(x^2-1)(x-2)=0$$

$\Rightarrow \quad x^2=1$ or $x=2$

$\Rightarrow \quad x=\pm 1$ or 2

$\Rightarrow \quad x=-1, 1, 2$

Thus, the given curve cuts the X-axis at points $(-1, 0)$, $(1, 0)$ and $(2, 0)$. **(1)**

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx}=(x^2-1)\cdot 1+(x-2)\cdot 2x$$

[by product rule]

$$\Rightarrow \quad \frac{dy}{dx}=x^2-1+2x^2-4x$$

$$\Rightarrow \quad \frac{dy}{dx}=3x^2-4x-1$$ **(1)**

12. Find the equation of tangent to the curve $4x^2+9y^2=36$ at the point $(3\cos\theta, 2\sin\theta)$.

Delhi 2011C

Given equation of curve is

$$4x^2 + 9y^2 = 36$$

On differentiating both sides w.r.t. x, we get

$$8x + 18y\frac{dy}{dx} = 0 \Rightarrow 18y\frac{dy}{dx} = -8x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{8x}{18y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-4x}{9y} \quad \text{...(i) } \mathbf{(1)}$$

But given that, tangent passes through the point $(3\cos\theta, 2\sin\theta)$.

On putting $x = 3\cos\theta$, $y = 2\sin\theta$ in Eq. (i), we get

$$\frac{dy}{dx} = \frac{-12\cos\theta}{18\sin\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2\cos\theta}{3\sin\theta}$$

$\therefore$ Slope of the tangent, $m = \dfrac{-2 \cos \theta}{3 \sin \theta}$ **(1)**

$$\left\{ \because m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} \right\}$$

Now, equation of tangent at the point $(3 \cos \theta, 2 \sin \theta)$ having slope

$$m = -\frac{2 \cos \theta}{3 \sin \theta} \text{ is}$$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - 2 \sin \theta = \frac{-2 \cos \theta}{3 \sin \theta}(x - 3 \cos \theta) \quad \textbf{(1)}$$

$$\Rightarrow 3y \sin \theta - 6 \sin^2 \theta = -2x \cos \theta + 6 \cos^2 \theta$$

$$\Rightarrow 2x \cos \theta + 3y \sin \theta - 6(\sin^2 \theta + \cos^2 \theta) = 0$$

$$\Rightarrow 2x \cos \theta + 3y \sin \theta - 6 = 0$$

$$[\because \sin^2 \theta + \cos^2 \theta = 1]$$

which is the required equation of tangent. **(1)**

13. Find the equation of tangent to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at point $x = 1, y = 0$. **Delhi 2011C**

Given equation of curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10 \quad \textbf{(1)}$$

Slope of a tangent at point (1,0) is

$$m = \left[\frac{dy}{dx}\right]_{x=1} = 4 - 18 + 26 - 10 = 2 \quad \textbf{(1)}$$

$\therefore$ Equation of tangent at point (1,0) having slope 2 is **(1)**

$$y - 0 = 2(x - 1)$$

$$\Rightarrow \quad y = 2x - 2$$

Hence, required equation of tangent is $2x - y = 2$. **(1)**

14. Find the values of x for which $f(x) = [x(x-2)]^2$ is an increasing function. Also, find the points on the curve, where the tangent is parallel to X-axis.

Delhi 2010

(i) Firstly, differentiate the given function with respect to x and put $f'(x) = 0$, then find the value of x and check the interval in which $f'(x) \geq 0$.

(ii) The tangent is parallel to X-axis, i.e. $\frac{dy}{dx} = 0$ get different values of x and put in the given curve to get corresponding values of y.

The given function is $f(x) = [x(x-2)]^2$.

$$\Rightarrow \quad f(x) = (x^2 - 2x)^2 \quad \ldots \text{(i)}$$

We have to find the values of x for which $f(x)$ is an increasing function.

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$f'(x) = 2(x^2 - 2x)\frac{d}{dx}(x^2 - 2x)$$

$$= 2(x^2 - 2x)(2x - 2)$$

On putting $f'(x) = 0$, we get

$$2(x^2 - 2x)(2x - 2) = 0$$

$$\Rightarrow \quad 4x(x-2)(x-1) = 0$$

$$\Rightarrow \quad 4x = 0 \text{ or } x - 2 = 0 \text{ or } x - 1 = 0$$

$$\Rightarrow \quad x = 0, 1, 2 \qquad \text{...(ii)}\textbf{(1)}$$

Now, we find the intervals in which $f(x)$ is an increasing function.

Interval	$f'(x) = 4x(x-1)(x-2)$	Sign of $f'(x)$
$x<0$	(−)(−)(−)	−ve
$0<x<1$	(+)(−)(−)	+ve
$1<x<2$	(+)(+)(−)	−ve
$x>2$	(+)(+)(+)	+ve

(1)

From the above table, it is clear that the given function $f(x)$ is an increasing function when $0<x<1$ and when $x>2$. Because at these values of x, $f'(x)$ is positive and we know that, $f(x)$ is said to be an increasing function whenever $f'(x) \geq 0$. **(1)**

Also, we have to find the points on the given curve where the tangent is parallel to X-axis. We know that, when a tangent is parallel to X-axis, then

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 2(x^2 - 2x)(2x - 2) = 0$$

$$\Rightarrow \quad x = 0, 1, 2$$

When $x = 0$, then $y = [0(-2)]^2 = 0$

When $x = 1$, then $y = [1(-1)]^2 = 1$

When $x = 2$, then $y = [2(0)]^2 = 0$

Hence, the tangent is parallel to X-axis at the points (0, 0), (1, 1) and (2, 0). **(1)**

15. Find the equation of tangent to the curve $y = \frac{x-7}{x^2-5x+6}$ at the point, where it cuts the X-axis. **All India 2010C, 2010**

Given equation of curve is

$$y = \frac{x-7}{x^2-5x+6} \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2-5x+6)\cdot 1-(x-7)(2x-5)}{(x^2-5x+6)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[(x^2-5x+6) - y(x^2-5x+6)(2x-5)\right]}{(x^2-5x+6)^2}$$

$$\left[\because \text{given, } y = \frac{x-7}{x^2-5x+6} \quad \therefore (x-7) = y(x^2-5x+6)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-(2x-5)y}{x^2-5x+6} \quad \text{...(ii)}$$

[dividing numerator and denominator by x^2-5x+6] **(1)**

Also, given that given curve cuts X-axis, so its y-coordinate is zero.

$\therefore$ Put $y = 0$ in Eq. (i), we get

$$\frac{x-7}{x^2-5x+6} = 0$$

$$\Rightarrow \quad x = 7 \quad \textbf{(1)}$$

So, curve passes through the point (7, 0).

Now, slope of tangent at (7,0) $= m = \left[\frac{dy}{dx}\right]_{(7,\,0)}$

$$= \frac{1-0}{49-35+6} = \frac{1}{20} \qquad \textbf{(1)}$$

Hence, the required equation of tangent passing through the point (7, 0) having slope 1/20 is

$$y - 0 = \frac{1}{20}(x-7) \Rightarrow 20y = x - 7$$

$$\Rightarrow \quad x - 20y = 7 \qquad \textbf{(1)}$$

16. Find the equations of the normal to the curve $y = x^3 + 2x + 6$, which are parallel to line $x + 14y + 4 = 0$. **HOTS; Delhi 2010**

> We have to find equation of normal which is parallel to line $x + 14y + 4 = 0$. Here, slope of normal is equal to that of given line. So, first find slope $\frac{-1}{\left(\frac{dy}{dx}\right)}$ of the normal and equate it to slope of the given line. Then, find the required equation.

Given equation of curve is

$$y = x^3 + 2x + 6 \qquad \text{...(i)}$$

and the given equation of line is

$$x + 14y + 4 = 0$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 + 2$$

$$\therefore \text{Slope of normal} = \frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-1}{3x^2+2}$$

Also, slope of the line $x+14y+4=0$ is $-\frac{1}{14}$.

(1)

$\left[\because \text{ slope of the line } Ax+By+C=0 \text{ is} -\frac{A}{B}.\right]$

[$\because$ we know that, if two lines are parallel, then their slopes are equal]

$$\therefore \quad 3x^2+2=14$$

$$\Rightarrow \quad 3x^2=12 \Rightarrow x^2=4$$

$$\Rightarrow \quad x=\pm 2$$ **(1)**

From Eq. (i), when $x=2$, then

$$y=(2)^3+2(2)+6$$
$$=8+4+6=18$$

and when $x=-2$, then

$$y=(-2)^3+2(-2)+6$$
$$=-8-4+6=-6$$

$\therefore$ Normal passes through (2, 18) and (–2, –6).

Also, slope of normal $=\frac{-1}{14}$.

Hence, equation of normal at point (2, 18) is

$$y-18=\frac{-1}{14}(x-2)$$

$$\Rightarrow \quad 14y-252=-x+2$$

$$\Rightarrow \quad x+14y=254$$ **(1)**

and equation of normal at point (–2, –6) is

$$y+6=-\frac{1}{14}(x+2)$$

$$\Rightarrow \quad 14y+84=-x-2$$

$$\Rightarrow \quad x+14y=-86$$ **(1)**

Hence, the two equations of normal are $x+14y=254$ and $x+14y=-86$.

17. Find the equation of tangent to the curve $x^2 + 3y = 3$, which is parallel to line $y - 4x + 5 = 0$. **Delhi 2009C**

Given equation of the curve is

$$x^2 + 3y = 3 \quad \text{...(i)} \quad \textbf{(1/2)}$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x + 3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{3}$$

$\therefore$ Slope (m) of tangent $= -\dfrac{2x}{3}$ **(1/2)**

Given equation of the line is

$$y - 4x + 5 = 0 \Rightarrow y = 4x - 5$$

which is of the form $y = mx + c$.

$\therefore$ Slope of the line is $m = 4$.

Now, tangent is parallel to the given line.

$\therefore$ Slope of tangent = Slope of line

$$\Rightarrow \quad -\frac{2x}{3} = 4 \Rightarrow -2x = 12$$

$$\Rightarrow \quad x = -6 \quad \textbf{(1)}$$

On putting $x = -6$ in Eq. (i), we get

$$(-6)^2 + 3y = 3 \Rightarrow 3y = 3 - 36$$

$$\Rightarrow \quad 3y = -33$$

$$y = -11 \quad \textbf{(1)}$$

So, the tangent is passing through point $(-6, -11)$ and it has slope 4.

Hence, the required equation of tangent is

$$y + 11 = 4(x + 6) \Rightarrow y + 11 = 4x + 24$$

$$\Rightarrow 4x - y = -13 \quad \textbf{(1)}$$

18. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line $4x - 2y + 5 = 0$. **Delhi 2009**

Do same as Que. 17. **[Ans.** $48x - 24y = 23$**]**

19. At what points will the tangent to the curve $y = 2x^3 - 15x^2 + 36x - 21$ be parallel to X-axis? Also, find the equations of tangents to the curve.

Given, equation of curve is

$y = 2x^3 - 15x^2 + 36x - 21$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x^2 - 30x + 36 \quad \textbf{(1)}$$

On puttting $\frac{dy}{dx} = 0$, we get

$6x^2 - 30x + 36 = 0 \Rightarrow 6(x^2 - 5x + 6) = 0$

$\Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$

$\Rightarrow x = 2, 3$ **(1)**

Now, when $x = 2$, then from Eq. (i), we get

$y = 2(2)^3 - 15(2)^2 + 36(2) - 21$

$\Rightarrow y = 16 - 60 + 72 - 21$

$= 88 - 81 = 7$

Also, when $x = 3$, then from Eq. (i), we get

$$y = 2(3)^3 - 15(3)^2 + 36(3) - 21$$
$$= 54 - 135 + 108 - 21$$
$$\Rightarrow \quad y = 162 - 156 = 6$$

Hence, the tangent passes through the points (2, 7) and (3, 6).

Now, we find the equation of tangents to the given curve.

$\therefore$ Slope (m_1) of tangent at point (2, 7) is

$$m_1 = \left[\frac{dy}{dx}\right]_{(2,7)} = 6(2)^2 - 30(2) + 36$$
$$= 24 - 60 + 36 = 0 \qquad \textbf{(1)}$$

and slope (m_2) of tangent at point (3, 6) is

$$m_2 = \left[\frac{dy}{dx}\right]_{(3,6)} = 6(3)^2 - 30(3) + 36$$
$$= 54 - 90 + 36 = 0$$

$\therefore$ Equation of tangent at point (2, 7) having slope 0 is

$$y - 7 = 0(x - 2) \Rightarrow y - 7 = 0 \Rightarrow y = 7$$

and equation of tangent at point (3, 6) having slope 0 is

$$y - 6 = 0(x - 3)$$
$$\Rightarrow \quad y - 6 = 0 \Rightarrow y = 6 \qquad \textbf{(1)}$$

Hence, equation of tangents are $y = 7$ and $y = 6$.

6 marks Questions

20. Find the equations of the tangent to the curves $y = x^2 - 2x + 7$ which is

(i) parallel to the line $2x - y + 9 = 0$,

(ii) perpendicular to the line $5y - 15x = 13$. **Delhi 2014C**

Given equation of curve is

$$y = x^2 - 2x + 7 \quad \text{...(i)}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2x - 2 \quad \textbf{(1)}$$

(i) The equation of the line is $2x - y + 9 = 0$

$$\Rightarrow \quad y = 2x + 9$$

which is of the form $y = mx + c$

$\therefore$ Slope of the line is $m = 2$

If a tangent is parallel to the line, then slope of tangent is equal to the slope of the line.

Therefore, $\frac{dy}{dx} = m$

$$\Rightarrow \quad 2x - 2 = 2$$

$$\Rightarrow \quad x = 2$$

When $x = 2$, then from Eq. (i), we get

$$y = 2^2 - 2 \times 2 + 7$$

$$\Rightarrow \quad y = 7 \quad \textbf{(1)}$$

The point on the given curve at which tangent is parallel to given line is (2, 7) and the equation of the tangent is

$$y - 7 = 2(x - 2)$$

$$\Rightarrow \quad 2x - y + 3 = 0$$

Hence, the equation of the tangent line to the given curve which is parallel to line $2x - y + 9 = 0$ is $y - 2x - 3 = 0$. **(1)**

(ii) The equation of the given line is

$$5y - 15x = 13$$

$$\Rightarrow \quad y = \frac{15x + 13}{5} = 3x + \frac{13}{5}$$

which is of the from $y = mx + c$.

$\therefore$ Slope of the given line is 3.

If a tangent is perpendicular to the line $5y - 15x = 13$.

Then, the slope of the tangent $= -\frac{1}{3}$

$\therefore 2x - 2 = \frac{-1}{3} \Rightarrow x = \frac{5}{6}$

When $x = \frac{5}{6}$, then from Eq. (i), we get

$$y = \left(\frac{5}{6}\right)^2 - 2\left(\frac{5}{6}\right) + 7$$

$$\Rightarrow \quad y = \frac{217}{36} \qquad \textbf{(1)}$$

$\therefore$ The point on the given curve at which tangent is perpendicular to given line is $\left(\frac{5}{6}, \frac{217}{36}\right)$ and the equation of the tangent is

$$y - \frac{217}{36} = \frac{-1}{3}\left(x - \frac{5}{6}\right)$$

$$\Rightarrow \quad \frac{36y - 217}{36} = \frac{-x}{3} + \frac{5}{18}$$

$$\Rightarrow \quad 12x + 36y - 227 = 0 \qquad \textbf{(1)}$$

Hence, the equation of the tangent line to the given curve which is perpendicular to the line $5y - 15x = 13$ is
$36y + 12x - 227 = 0$. **(1)**

21. Find the equation of the normal at a point on the curve $x^2 = 4y$, which passes through the point (1, 2). Also, find the equation of the corresponding tangent.
Delhi 2013

Given curve is $x^2 = 4y$

On differentiating both sides w.r.t. x, we get

$$2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \qquad \textbf{(1)}$$

Let (h, k) be the coordinates of the point of contact of the normal to the curve $x^2 = 4y$. Then, slope of the tangent at (h, k) is given by

$$\left[\frac{dy}{dx}\right]_{(h,k)} = \frac{h}{2}$$

and slope of the normal at $(h, k) = \frac{-2}{h}$

Therefore, the equation of normal at (h, k) is

$$y - k = \frac{-2}{h}(x - h) \quad \text{...(i) } \mathbf{(1)}$$

$$\left[\begin{array}{l}\because \text{ equation of normal in slope} \\ \text{form is } y - y_1 = -\frac{1}{m}(x - x_1)\end{array}\right]$$

Since, it passes through the point (1, 2), so on putting $x = 1$ and $y = 1$, we get

$$2 - k = \frac{-2}{h}(1 - h)$$

$$\Rightarrow \quad k = 2 + \frac{2}{h}(1 - h) \quad \text{...(ii) } \mathbf{(1)}$$

On since, (h, k) also lies on the curve $x^2 = 4y$, so

$$h^2 = 4k \quad \text{...(iii) } \mathbf{(1)}$$

On solving Eqs.(ii) and (iii), we get
$h = 2$ and $k = 1$.

Substituting the values of h and k in Eq.(i), the required equation of normal is

$$y - 1 = \frac{-2}{2}(x - 2) \Rightarrow x + y = 3 \ \mathbf{(1)}$$

Now, equation of tangent at (h, k) is

$$y - k = \frac{h}{2}(x - h)$$

On putting $h = 2$ and $k = 1$, we get

$$y - 1 = \frac{2}{2}(x - 1) \Rightarrow y - 1 = x - 2$$

$$\Rightarrow \quad y = x - 1 \quad \mathbf{(1)}$$

22. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3}, 0\right)$.

HOTS; All India 2013

> Firstly, differentiate the given curve with respect to x and determine $\frac{dy}{dx}$. Then, find the equation of tangent at (x_1, y_1) and since it passes through given point (x_0, y_0), so this point will satisfy the tangent and curve also.

Given equation of curve is

$$3x^2 - y^2 = 8 \quad \ldots(i)$$

On differentiating both sides w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3x}{y} \qquad \textbf{(1)}$$

Equation of tangent at point (h, k) is

$$y - k = \left(\frac{dy}{dx}\right)_{(h,\,k)}(x - h)$$

$$\Rightarrow \qquad y - k = \frac{3h}{k}(x - h) \qquad \text{... (ii)} \ \textbf{(1)}$$

Since, it is passes through the point $\left(\frac{4}{3}, 0\right)$.

$$\therefore \quad 0 - k = \frac{3h}{k}\left(\frac{4}{3} - h\right) \Rightarrow -k^2 = 3h\frac{(4 - 3h)}{3}$$

$$\Rightarrow 3h^2 - k^2 - 4h = 0 \qquad \text{...(iii)} \ \textbf{(1)}$$

Also, the point (h, k) satisfy the Eq. (i), so we get

$$3h^2 - k^2 = 8 \qquad \text{...(iv)}$$

Now, on solving Eqs. (iii) and (iv), we get

$$4h = 8 \Rightarrow h = 2$$

On putting $h = 2$ in Eq. (iv), we get

$$3(2)^2 - k^2 = 8 \Rightarrow k^2 = 4$$

$$\Rightarrow \qquad k = \pm 2 \qquad \textbf{(1)}$$

Now, putting the values of h and k in Eq. (ii), we get

$$y - (\pm 2) = \frac{3(2)}{\pm 2}(x - 2)$$

$$\Rightarrow \quad y \mp 2 = \pm 3(x - 2) \Rightarrow y = \pm 3x \mp 6 \pm 2$$

$$\Rightarrow \qquad y = \pm 3x \mp 4 \qquad \textbf{(1)}$$

It will gives four possible equations but out of them only $y = -3x + 4$ and $y = +3x - 4$ satisfies the point $\left(\frac{4}{3}, 0\right)$. Hence, there are two required equations of tangent. **(1)**

23. For the curve $y = 4x^3 - 2x^5$, find all the points on the curve at which the tangent passes through the origin. **Delhi 2013C**

Given curve is $y = 4x^3 - 2x^5$ ...(i)

Let any point on the curve is (x_1, y_1).

$\therefore \quad y_1 = 4x_1^3 - 2x_1^5$ ...(ii)

On differentiating both sides of Eq. (i), we get

$$\frac{dy}{dx} = 12x^2 - 10x^4 \quad \textbf{(1/2)}$$

Equation of tangent at point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$\Rightarrow \quad y - y_1 = [12(x_1)^2 - 10(x_1)^4](x - x_1)$ **(1)**

Since, it passes through the origin.

$\therefore \quad 0 - y_1 = (12x_1^2 - 10x_1^4)(0 - x_1)$

$\Rightarrow \quad y_1 = (12x_1^2 - 10x_1^4)x_1$ ...(iii) **(1/2)**

From Eqs. (ii) and (iii), we get

$(12x_1^2 - 10x_1^4)x_1 = 4x_1^3 - 2x_1^5$

$\Rightarrow 2x_1^3(6 - 5x_1^2) = 2x_1^3(2 - x_1^2)$

$\Rightarrow 2x_1^3(4 - 4x_1^2) = 0 \Rightarrow x_1 = 0$ or $4 - 4x_1^2 = 0$

$\Rightarrow \quad x_1 = 0$ or $x_1 = \pm 1$ **(1)**

On putting the values of $x_1 = 0$, 1 and -1 respectively in Eq. (i), we get

At $x_1 = 0 \Rightarrow y_1 = 0$

At $x_1 = 1$

$\Rightarrow \quad y_1 = 4(1)^3 - 2(1)^5 = 4 - 2 = 2$ **(1)**

and at $x_1 = (-1)$, $y_1 = 4(-1)^3 - 2(-1)^5$

$= 4(-1) - 2(-1) = -4 + 2 = -2$ **(1)**

Hence, all points on the curve at which the tangent passes through origin are $(0, 0)$, $(1, 2)$ and $(-1, -2)$. **(1)**

24. Find the equations of tangent and normal to the curve $x = 1 - \cos\theta, y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$. All India 2010

Firstly, differentiate the given curve with respect to θ and then determine $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$. Further, use the formula, equation of tangent at (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

and equation of normal at (x_1, y_1) is

$y - y_1 = -\frac{1}{m}(x - x_1)$.

Given curve is $x = 1 - \cos\theta$ and $y = \theta - \sin\theta$.

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(1 - \cos\theta) = \sin\theta$$

and $$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta - \sin\theta) = 1 - \cos\theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

At $\theta = \frac{\pi}{4}$, $$\left[\frac{dy}{dx}\right]_{\theta = \frac{\pi}{4}} = \frac{1 - \cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1 - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2} - 1$$ **(1)**

Also, at $\theta = \frac{\pi}{4}, x_1 = 1 - \cos\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$

and $$y_1 = \frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{\sqrt{2}}$$ **(1)**

Now, we know that equation of tangent at (x,y) having slope m is given by

$$y - y_1 = m(x - x_1)$$

$$\therefore\ y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = (\sqrt{2} - 1)\left[x - \left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right)\right]$$

$$\Rightarrow\ y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = x(\sqrt{2} - 1) - \frac{(\sqrt{2} - 1)^2}{\sqrt{2}}$$ **(1)**

4 $\sqrt{2}$ $\sqrt{2}$

$$\Rightarrow \quad y-\frac{\pi}{4}+\frac{1}{\sqrt{2}}=x(\sqrt{2}-1)-\frac{(2+1-2\sqrt{2})}{\sqrt{2}}$$

$$[\because (a-b)^2=a^2+b^2-2ab]$$

$$\Rightarrow \left(y-\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right)=x(\sqrt{2}-1)-\frac{(3-2\sqrt{2})}{\sqrt{2}}$$

$$\Rightarrow \quad x(\sqrt{2}-1)-y=\frac{3-2\sqrt{2}}{\sqrt{2}}-\frac{\pi}{4}+\frac{1}{\sqrt{2}}$$

Hence, the equation of tangent is

$$x(\sqrt{2}-1)-y=\frac{12-8\sqrt{2}-\sqrt{2}\pi+4}{4\sqrt{2}}$$

$$\Rightarrow x(8-4\sqrt{2})-4\sqrt{2}y=(16-\sqrt{2}\pi-8\sqrt{2}) \quad \textbf{(1)}$$

Also, the equation of normal at (x_1, y_1) having slope $-\frac{1}{m}$ is given by

$$y-y_1=-\frac{1}{m}(x-x_1)$$

$$\Rightarrow y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right)=\frac{-1}{\sqrt{2}-1}\left(x-\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y(\sqrt{2}-1)-\left(\frac{\sqrt{2}\pi-4}{4\sqrt{2}}\right)(\sqrt{2}-1)$$

$$=-x+\frac{\sqrt{2}-1}{\sqrt{2}} \qquad \textbf{(1)}$$

$$\Rightarrow \quad y(\sqrt{2}-1)-\left(\frac{2\pi-\sqrt{2}\pi-4\sqrt{2}+4}{4\sqrt{2}}\right)$$

$$=\frac{-\sqrt{2}x+\sqrt{2}-1}{\sqrt{2}}$$

$$\Rightarrow \quad 4\sqrt{2}y(\sqrt{2}-1)-2\pi+\sqrt{2}\pi+4\sqrt{2}-4$$

$$=-4\sqrt{2}x+4\sqrt{2}-4$$

$$\Rightarrow \quad 4\sqrt{2}x + y(8 - 4\sqrt{2}) = 2\pi - \sqrt{2}\pi$$

$$\Rightarrow \quad 4\sqrt{2}x + (8 - 4\sqrt{2})y = \pi(2 - \sqrt{2}) \qquad \textbf{(1)}$$